REPORT NO. RS-TR-65-1

SOLUTION OF TRANSIENT HEAT TRANSFER PROBLEMS FOR FLAT PLATES, CYLINDERS, AND SPHERES BY FINITE-DIFFERENCE METHODS

by

W. G. Burleson
R. Eppes, Jr.

March 1965

U S ARMY MISSILE COMMAND
REDSTONE ARSENAL, ALABAMA

FOR ERRATA

AD 461662

THE FOLLOWING PAGES ARE CHANGES

TO BASIC DOCUMENT

HEADQUARTERS
U. S. ARMY MISSILE COMMAND
Redstone Arsenal, Alabama

In Reply
Refer To AMSMI-RAP

22 June 1965

SUBJECT: Errata for Report No. RS-TR-65-1, Subject: Solution of Transient Heat Transfer Problems for Flat Plates, Cylinders, and Spheres by Finite-Difference Methods, dated 15 March 1965

TO: Recipients of Subject Report

It is requested that the following changes be made in your copy of the subject report.

1. Equation 15, page 8 should be:

$$\frac{k_a}{\tau_a}\left(R - \Sigma\tau_n + \frac{\tau_a}{2}\right)(T_{n-1} - T_n) + \frac{k_b}{\tau_b}\left(R - \Sigma\tau_n - \frac{\tau_b}{2}\right)(T_{n+1} - T_n) =$$

$$\left[\left(R - \Sigma\tau_n + \frac{\tau_a}{4}\right)(\rho_a C_a \tau_a) + \left(R - \Sigma\tau_n - \frac{\tau_b}{4}\right)(\rho_b C_b \tau_b)\right]\left[\frac{T'_n - T_n}{2\Delta t}\right]$$

2. Equation 16, page 8 should be

$$T'_n = T_n + \frac{2k_a \Delta t\left(R - \Sigma\tau_n + \frac{\tau_a}{2}\right)}{\tau_a\left[\left(R - \Sigma\tau_n + \frac{\tau_a}{4}\right)(\rho_a C_a \tau_a) + \left(R - \Sigma\tau_n - \frac{\tau_b}{4}\right)(\rho_b C_b \tau_b)\right]}(T_{n-1} - T_n) +$$

$$\frac{2k_b \Delta t\left(R - \Sigma\tau_n - \frac{\tau_b}{2}\right)}{\tau_b\left[\left(R - \Sigma\tau_n + \frac{\tau_a}{4}\right)(\rho_a C_a \tau_a) + \left(R - \Sigma\tau_n - \frac{\tau_b}{4}\right)(\rho_b C_b \tau_b)\right]}(T_{n+1} - T_n)$$

3. Replace four of the m's by G's in equation 33, page 17, so that the equation reads as follows:

$$T_n' = T_n + \left(T_{n-1} - T_n\right)\left\{\frac{2\Delta t k_a\left\{\left[R - \Sigma\tau_n + \frac{\tau_a}{2}(G - 1)\right]^m + \frac{\tau_a{}^m}{12}(m - 1)\right\}}{\tau_a\left\{\rho_a C_a \tau_a\left\{\left[R - \Sigma\tau_n + \frac{\tau_a}{4}(G - 1)\right]^m + \frac{\tau_a{}^m}{48}(m - 1)\right\} + \rho_b C_b \tau_b\left\{\left[R - \Sigma\tau_n - \frac{\tau_b}{4}(G - 1)\right]^m + \frac{\tau_b{}^m}{48}(m - 1)\right\}\right\}}\right\} +$$

$$\left(T_{n+1} - T_n\right)\left\{\frac{2\Delta t k_b\left\{\left[R - \Sigma\tau_n - \frac{\tau_b}{2}(G - 1)\right]^m + \frac{\tau_b{}^m}{12}(m - 1)\right\}}{\tau_b\left\{\rho_a C_a \tau_a\left\{\left[R - \Sigma\tau_n + \frac{\tau_a}{4}(G - 1)\right]^m + \frac{\tau_a{}^m}{48}(m - 1)\right\} + \rho_b C_b \tau_b\left\{\left[R - \Sigma\tau_n - \frac{\tau_b}{4}(G - 1)\right]^m + \frac{\tau_b{}^m}{48}(m - 1)\right\}\right\}}\right\}$$

4. **Equation 25, page 13 should be:**

$$\varphi \frac{k_a}{\tau_a} (T_{n-1} - T_n) \left[\left(R - \Sigma \tau_n + \frac{\tau_a}{2} \right)^2 + \frac{\tau_a^2}{12} \right] + \varphi \frac{k_b}{\tau_b} (T_{n+1} - T_n) \left[\left(R - \Sigma \tau_n - \frac{\tau_b}{2} \right)^2 + \frac{\tau_b^2}{12} \right] =$$

$$\varphi \frac{(T_n' - T_n)}{2\Delta t} \left\{ \rho_a C_a \tau_a \left[\left(R - \Sigma \tau_n + \frac{\tau_a}{4} \right)^2 + \frac{\tau_a^2}{48} \right] + \rho_b C_b \tau_b \left[\left(R - \Sigma \tau_n - \frac{\tau_b}{4} \right)^2 + \frac{\tau_b^2}{48} \right] \right\}$$

W. G. BURLESON
Stress & Thermo Anal Branch
Struct' res & Mechanics Lab, DR'D

DDC AVAILABILITY NOTICE

Qualified requesters may obtain copies of this report from DDC.

DISPOSITION INSTRUCTIONS

Destroy this report when it is no longer needed. Do not return it to the originator.

DISCLAIMER

The findings in this report are not to be construed as an official Department of the Army position, unless so designated by other authorized documents.

15 March 1965 REPORT NO. RS-TR-65-1

SOLUTION OF TRANSIENT HEAT TRANSFER PROBLEMS FOR FLAT PLATES, CYLINDERS, AND SPHERES BY FINITE-DIFFERENCE METHODS

by

W. G. Burleson

R. Eppes, Jr.

DA Project No. 1A013001A039

AMC Management Structure Code No. 501611844

Stress and Thermodynamics Analysis Branch

Structures and Mechanics Laboratory

Directorate of Research and Development

U. S. Army Missile Command

Redstone Arsenal, Alabama

ABSTRACT

Presented in this report are finite-difference (forward) heat transfer equations applicable to transient, radial heat flow in spheres and cylinders and applicable to transient, one-dimensional heat flow in flat plates. By very minor manipulations, a single heat transfer procedure can easily be utilized to determine transient heat flow in all three basic structure configurations.

For each skin configuration the accuracy of the finite-difference procedure, compared with exact analytical methods, depends on optimum selection of the calculation time increment and the incremental distance between temperature nodes in relation to the material thermal properties and on the closeness of the approximate temperature gradients to the true gradients.

TABLE OF CONTENTS

LIST OF SYMBOLS

A -, area

θ - angle in radians

φ - solid angle in steradians

β - dimensionless modulus

R - radius of cylindrical or spherical section

$\Sigma\tau_n$ - distance from outer surface of cylindrical or spherical section to temperature point n

ϵ - surface emissivity

Δt - time increment for computation

T - temperature

k - thermal conductivity of material

C - specific heat of material

ρ - density of material

h - heat transfer coefficient

τ - incremental thickness for each material

T_r - radiation sink temperature

T_{eff} - effective gas temperature (boundary layer)

q_{conv} - convective heating

q_{rad} - radiative heating

q_{solar} - solar heating

q_{cond} - heat conducted

q_{stored} - stored energy

σ - Stefan-Boltzmann constant

LIST OF SYMBOLS (Concluded)

Subscripts or Superscripts

a - material "A"

b - material "B"

bs - backside or internal surface

o - external surface

i - internal surface

r - radiation sink

' - condition existing arter the lapse of one (1) Δt

n - nodal point

x - nodal point

Section I. INTRODUCTION

The U. S. Army has had great success using finite-difference (forward) techniques to accurately determine transient heat transfer in homogeneous and composite wall missile structures. Heating effects on most Army missile structures have been analyzed by use of finite-difference methods applicable to transient, one-dimensional heat flow in flat plates. This flat-plate technique is generally quite adequate for structure analyses of missiles such as JUPITER, PERSHING, REDSTONE, and SERGEANT because most of the airframe exposed to inflight aerodynamic heating has a relatively large radius of curvature in relation to the skin thickness. Thus the curved structure actually approaches a flat plate in relation to heat storage and heat flow.

Contrarily, a flat-plate solution may not suffice for analysis of small hemispherical tips, small hemicylindrical leading edges, small caliber vehicles, large or small cylindrical and spherical solid propellant motors exposed to environmental thermal shocks, and many other missile components and sections. Thus, accurate, flexible procedures for at least three basic structure configurations are necessary for proper thermal analyzation of present and future Army missile systems.

One purpose of this report is to focus attention on the similarities existing in finite-difference heat transfer equations for radial heat flow in spheres and cylinders and one-dimensional heat flow in flat plates. Another purpose of this report is to present one general heat transfer procedure which can be used to determine transient heat transfer effects in three basic configurations, i.e., sphere, cylinder, and flat plate.

Section II. FLAT-PLATE, CYLINDER, AND SPHERE PROCEDURES

1. Background

Transient, finite-difference (forward) heat transfer methods have occupied a vital role in the development of several of the Army's most prominent, reliable missile systems. During the historical heat protection material development program for the JUPITER nose cone, flat-plate, finite-difference, heat transfer methods were used successfully in analyses of reentry simulation test results[1] and in analyses of inflight aerodynamic heating effects.[2,3] This same heat transfer method, in conjunction with a simultaneous ablation routine,[4] was used exclusively to determine the thermal protection requirements for the PERSHING reentry body. Later, calculated inflight temperatures were found to be in good agreement with temperatures measured on the reentry body of PERSHING missile 308.[5] Agreement of finite-difference results with exact solutions is discussed briefly in Paragraph 6 of this section.

With the advent of large solid-propellant power plants for use in severe environments, as well as increased demand for hypersonic, low-altitude flights by small vehicles or large vehicles having small leading edges or tips, the Army has become increasingly involved with heat flow problems which are not adequately solvable by flat-plate techniques. Recently, therefore, transient, radial heat transfer procedures for cylinders and spheres, in finite-difference form, have been derived. Using the radial procedure for a cylinder, good agreement between measured and calculated temperatures of a PERSHING XB-1 motor subjected to severe thermal shock in a cooling chamber were obtained.[6]

Procedures for determining radial heat flow in cylinders and spheres have many applications in missile design and analysis, since in numerous cases the assumption of only radial flow is sufficiently accurate. For instance, transient, radial heat flow in a small, solid hemispherical tip may be adequate for materials with low thermal diffusivities (reinforced plastics, for example) even when the heat input varies with external surface station. Transient, radial heat flow at the midsection of a cylindrical solid-propellant motor having a large L/D ratio may be adequate in most cases because the present solid propellants have small thermal diffusivities and end effects do not influence the motor midsection for many hours.[6]

The conduction equations presented for each of the three basic structure configurations are applicable to any number of material layers; however, only two layers are illustrated because this is the maximum number of materials necessary for the application of all

equations for thermally thick skins. Equations for thermally thin skins and thick-thin combinations have been derived for flat plates, cylinders, and spheres to account for all practical material arrangements. These equations are not presented due to similarity of the derivations to those of the thick-thick wall cases discussed.

Many statements made throughout the discussion of the flat-plate procedure are also applicable to the other two procedures, and are not repeated to preclude repetition. Since the average area of a sphere across which heat flows is not a linear function of the radius, the derivation of average areas for the spherical configuration is shown in Appendix A. Shape factors have been purposely omitted from the radiation terms throughout the report.

2. Transient, One-Dimensional Heat Transfer for Flat Plates

One-dimensional, flat-plate heat transfer in a homogeneous material can be determined by solving heat balance equations at the exposed surface, unexposed surface, interior nodes, and interfaces. The forward finite difference method was used. It was assumed that the incremental thickness (τ) can be selected sufficiently small to give accurate temperature gradients between adjacent nodes and that the incremental time (Δt) is small enough to neglect any effect on regions more than one τ from the node in question.

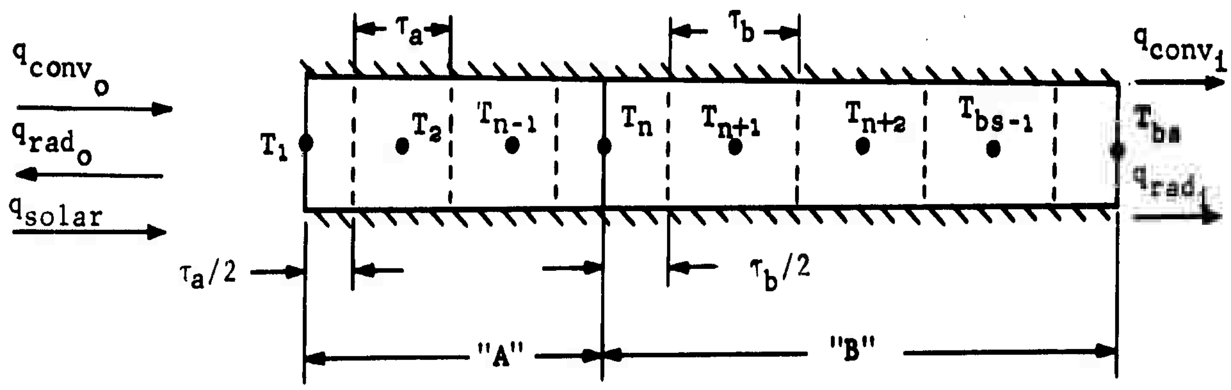

Figure 1

a. Exposed Surface

From Figure 1 the heat balance at the exposed surface is

$$q_{conv_o} + q_{solar} - q_{rad_o} - q_{cond_{1\rightarrow 2}} = q_{stored_1} \tag{1}$$

where

$$q_{conv_o} = h_o A\,(T_{eff_o} - T_1)$$

q_{solar} = heat received from sun per unit area

$$q_{rad_o} = \epsilon_o \sigma A\,(T_1^4 - T_{r_o}^4)$$

$$q_{\substack{cond \\ 1 \to 2}} = \frac{k_a A\,(T_1 - T_2)}{\tau_a}$$

$$q_{\substack{sto \\ 1}} = \rho_a C_a \frac{\tau_a}{2} \frac{A(T_1' - T_1)}{\Delta t}$$

The area, A, is uniform for one-dimensional, flat-plate heat transfer. Rewriting Equation (1) we have

$$h_o(T_{eff_o} - T_1) + q_{solar} + \epsilon_o \sigma(T_{r_o}^4 - T_1^4) + \frac{k_a(T_2 - T_1)}{\tau_a} = \rho_a C_a \frac{\tau_a}{2} \frac{(T_1' - T_1)}{\Delta t} \qquad (2)$$

Multiply by $\frac{2\Delta t}{\rho_a C_a \tau_a}$, let $\beta = \frac{k\Delta t}{\rho C \tau^2}$, let $\gamma = \frac{\Delta t}{\rho C \tau}$, and solve for T_1'

$$T_1' = T_1(1 - 2\gamma_a h_o - 2\beta_a) + 2\beta_a T_2 + 2\gamma_a h_o T_{eff_o} + 2\gamma_a \epsilon_o \sigma(T_{r_o}^4 - T_1^4) + 2\gamma_a q_{solar} \qquad (3)$$

The coefficient of T_1 in Equation (3) must be $\geqq 0$ otherwise T_1' will depend on T_1 in the negative sense which is not reasonable. Therefore, set the coefficient of T_1 equal to zero and solve for β_a. Note after Equation (2) that $\gamma = \beta(\tau/k)$.

$$1 - 2\gamma_a h_o - 2\beta_a = 0$$

$$\beta_a = \frac{1}{2 + \frac{2(h_o \tau_a)}{k_a}} \qquad (4)$$

This beta is denoted beta critical (β_a) and is the upper limit of the β in Equation (8) for convergence. It is readily seen that if β in Equation (8) is a maximum of 0.5 (to make the coefficient of $T_n = 0$) then the coefficient of T_1 in Equation (3) is negative because $2\gamma h$ is positive. Therefore, β must be equal to or less than $\frac{1}{2 + \frac{2(h_o \tau_a)}{k_a}}$

The maximum h_o expected or calculated for the specific case for analysis should be used in determining the critical β.

b. Interface Equation

At the interface between material "A" and "B" (T_n, Figure 1) the energy balance is

$$\underset{n-1 \rightarrow n}{q_{cond}} - \underset{n \rightarrow n+1}{q_{cond}} = \underset{n}{q_{stored}} \tag{5}$$

or

$$\frac{k_a(T_{n-1} - T_n)}{\tau_a} + \frac{k_b(T_{n+1} - T_n)}{\tau_b} = \frac{(\rho_a C_a \tau_a + \rho_b C_b \tau_b)(T_n' - T_n)}{2\Delta t}$$

Rearrange and solve for T_n'

$$T_n' = T_n + (T_{n-1} - T_n)\left[\frac{2k_a\Delta t}{\tau_a(\rho_a C_a \tau_a + \rho_b C_b \tau_b)}\right] + (T_{n+1} - T_n)\left[\frac{2k_b\Delta t}{\tau_b(\rho_a C_a \tau_a + \rho_b C_b \tau_b)}\right] \tag{6}$$

c. Internal Nodes

The heat balance at any interior point of a homogeneous wall (Figure 1) may be written

$$\underset{n \rightarrow n+1}{q_{cond}} - \underset{n+1 \rightarrow n+2}{q_{cond}} = \underset{n+1}{q_{stored}} \tag{7}$$

or

$$\frac{k_b(T_n - T_{n+1})}{\tau_b} + \frac{k_b(T_{n+2} - T_{n+1})}{\tau_b} = \rho_b C_b \tau_b \frac{(T_{n+1}' - T_{n+1})}{\Delta t}$$

Solving for T_{n+1}' with $\beta = \frac{k\Delta t}{\rho C \tau^2}$ gives

$$T_{n+1}' = T_{n+1}(1 - 2\beta_b) + \beta_b(T_n + T_{n+2}) \tag{8}$$

d. Backside Surface

The energy balance at the backside surface, T_{bs}, may be written as

$$q_{\underset{bs-1 \rightarrow bs}{cond}} - q_{conv_i} - q_{rad_i} = q_{\underset{bs}{stored}} \tag{9}$$

or

$$\frac{k_b(T_{bs-1} - T_{bs})}{\tau_b} + h_i(T_{eff_i} - T_{bs}) + \epsilon_i\sigma(T_{r_i}^4 - T_{bs}^4) = \frac{\rho_b C_b \tau_b(T'_{bs} - T_{bs})}{2\Delta t}$$

Rearrange and solve for T'_{bs}

$$T'_{bs} = T_{bs}(1 - 2\gamma_b h_i - 2\beta_b) + 2\beta_b T_{bs-1} + 2\gamma_b h_i T_{eff_i} + 2\gamma_b \epsilon_i \sigma(T_{r_i}^4 - T_{bs}^4) \tag{10}$$

To find the limiting β for the backside material, equate the coefficient of T_{bs} (Equation 10) to zero and solve for β_b.

$$\beta_b = \frac{1}{2 + \frac{2h_i\tau_b}{k_b}} \tag{11}$$

3. Transient, Radial Heat Transfer for Cylinders

a. External Surface Equation

Consider a cylindrical segment heated as shown in Figure 2.

From the energy balance at the peripheral surface

$$q_{conv_o} + q_{solar} - q_{rad_o} - q_{\underset{1 \rightarrow 2}{cond}} = q_{\underset{1}{stored}}$$

or

$$R\theta L\, h_o(T_{eff_o} - T_1) + R\theta L\, q_{solar} - R\theta L\, \epsilon_o\sigma\,(T_1^4 - T_{r_o}^4) -$$

$$\left(R - \frac{\tau_a}{2}\right)\theta L \frac{k_a}{\tau_a}(T_1 - T_2) = \left(R - \frac{\tau_a}{4}\right)\theta L\left(C_a\rho_a \frac{\tau_a}{2}\right)\frac{T_1' - T_1}{\Delta t}$$

(12)

Let $\beta = \dfrac{k\Delta t}{\rho C\tau^2}$ and $\gamma = \dfrac{\Delta t}{\rho C\tau}$, rearrange, and solve for T_1'.

$$T_1' = \left(1 - 2\beta_a \frac{1 - \frac{\tau_a}{2R}}{1 - \frac{\tau_a}{4R}} - 2\frac{\gamma_a h_o}{1 - \frac{\tau_a}{4R}}\right)T_1 + \left(2\beta_a \frac{1 - \frac{\tau_a}{2R}}{1 - \frac{\tau_a}{4R}}\right)T_2 +$$

$$\left(\frac{2\gamma_a h_o}{1 - \frac{\tau_a}{4R}}\right)T_{eff_o} + \frac{2\gamma_a \epsilon_o \sigma}{1 - \frac{\tau_a}{4R}}\left(T_{r_o}^4 - T_1^4\right) + \frac{2\gamma_a q_{solar}}{1 - \frac{\tau_a}{4R}}$$

(13)

Equating coefficient of T_1 to zero, since T_1' cannot depend on the previous T_1 in the negative sense;

$$0 = 1 - \frac{2}{1 - \frac{\tau_a}{4R}}\left[\gamma_a h_o + \beta_a\left(1 - \frac{\tau_a}{2R}\right)\right]$$

Since $\gamma = \beta\dfrac{\tau}{k}$, the critical or limiting β at the external surface is

$$\beta_a = \frac{1}{\left(\frac{2}{1 - \frac{\tau_a}{4R}}\right)\left(1 - \frac{\tau_a}{2R} + \frac{h_o\tau_a}{k_a}\right)} \qquad (14)$$

For $R \to \infty$ and τ finite or $\tau/R \ll 1$, Equation (14) approaches the one-dimensional flat plate case (Equation 4).

b. <u>Interface Equation</u> (Figure 2)

At point T_n (interface between Material "A" and "B") the energy balance becomes

$$q_{\substack{cond\\ n-1\to n}} - q_{\substack{cond\\ n\to n+1}} = q_{\substack{stored\\ n}} \qquad (15)$$

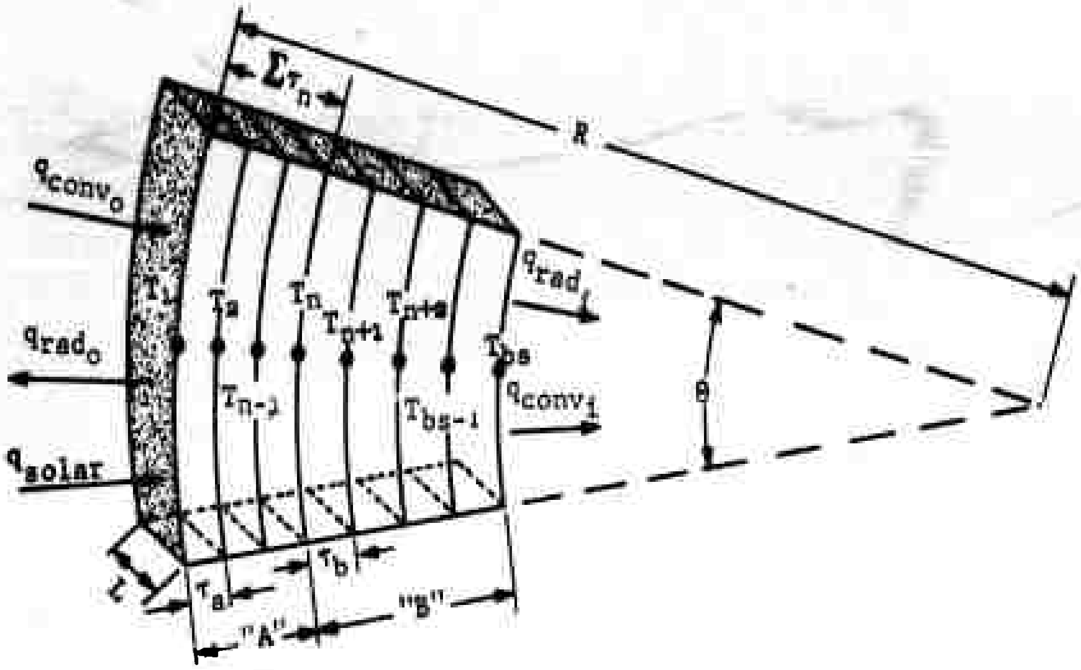

Figure 2

or

$$\frac{k_a}{\tau_a}\left(R - \Sigma\tau_n + \frac{\tau_a}{2}\right)(T_{n-1} - T_n) + \frac{k_b}{\tau_b}\left(R - \Sigma\tau_n - \frac{\tau_b}{2}\right)(T_{n+1} - T_n) =$$

$$(R - \Sigma\tau_n)\,\frac{(\rho_a C_a \tau_a + \rho_b C_b \tau_b)(T_n' - T_n)}{2\Delta t}$$

solving for T_n'

$$T_n' = T_n + \frac{2k_a\Delta t\left(R - \Sigma\tau_n + \frac{\tau_a}{2}\right)}{\tau_a(R - \Sigma\tau_n)(\rho_a C_a \tau_a + \rho_b C_b \tau_b)}(T_{n-1} - T_n) +$$

$$\frac{2k_b\Delta t\left(R - \Sigma\tau_n - \frac{\tau_b}{2}\right)}{\tau_b(R - \Sigma\tau_n)(\rho_a C_a \tau_a + \rho_b C_b \tau_b)}(T_{n+1} - T_n) \qquad (16)$$

c. <u>Interior Equation</u> (Figure 2)

At point T_{n+1} (Material "B") the energy balance becomes

$$\underset{n \to n+1}{q_{cond}} - \underset{n+1 \to n+2}{q_{cond}} = \underset{n+1}{q_{stored}} \qquad (17)$$

or

$$\frac{k_b}{\tau_b}\left(R - \Sigma\tau_{n+1} + \frac{\tau_b}{2}\right)(T_n - T_{n+1}) + \frac{k_b}{\tau_b}\left(R - \Sigma\tau_{n+1} - \frac{\tau_b}{2}\right)(T_{n+2} - T_{n+1}) =$$

$$(R - \Sigma\tau_{n+1})(\rho_b C_b \tau_b) \frac{(T'_{n+1} - T_{n+1})}{\Delta t}$$

Let $\beta = \frac{k\Delta t}{\rho C\tau^2}$ and $\gamma = \frac{\Delta t}{\rho C\tau}$

Solve for T'_{n+1}

$$T'_{n+1} = T_{n+1}\left\{1 - \beta_b\left[\frac{\left(R - \Sigma\tau_{n+1} + \frac{\tau_b}{2}\right) + \left(R - \Sigma\tau_{n+1} - \frac{\tau_b}{2}\right)}{R - \Sigma\tau_{n+1}}\right]\right\} + \beta_b\left\{T_n \frac{R - \Sigma\tau_{n+1} + \frac{\tau_b}{2}}{R - \Sigma\tau_{n+1}} + T_{n+2}\frac{R - \Sigma\tau_{n+1} - \frac{\tau_b}{2}}{R - \Sigma\tau_{n+1}}\right\} \quad (18)$$

d. Backside Surface Equation (Figure 2)

At point T_{bs} (Material "B") the energy balance becomes

$$q_{\substack{cond \\ bs-1 \to bs}} - q_{rad_i} - q_{conv_i} = q_{\substack{stored \\ bs}} \quad (19)$$

or

$$\frac{k_b}{\tau_b}\left(R - \Sigma\tau_{bs} + \frac{\tau_b}{2}\right)(T_{bs-1} - T_{bs}) + \epsilon_i\sigma(R - \Sigma\tau_{bs})(T^4_{r_i} - T^4_{bs}) +$$

$$h_i(R - \Sigma\tau_{bs})(T_{eff_i} - T_{bs}) = \left(\rho_b C_b \frac{\tau_b}{2}\right)\left(R - \Sigma\tau_{bs} + \frac{\tau_b}{4}\right)\left(\frac{T'_{bs} - T_{bs}}{\Delta t}\right)$$

Let $\beta = \frac{k\Delta t}{\rho C\tau^2}$ and $\gamma = \frac{\Delta t}{\rho C\tau}$

Solve for T'_{bs}

$$T'_{bs} = \left(1 - 2\beta_b \frac{R - \Sigma\tau_{bs} + \frac{\tau_b}{2}}{R - \Sigma\tau_{bs} + \frac{\tau_b}{4}} - 2\gamma_b h_i \frac{R - \Sigma\tau_{bs}}{R - \Sigma\tau_{bs} + \frac{\tau_b}{4}}\right) T_{bs} +$$

$$\left(2\beta_b \frac{R - \Sigma\tau_{bs} + \frac{\tau_b}{2}}{R - \Sigma\tau_{bs} + \frac{\tau_b}{4}}\right) T_{bs-1} + \left(2\gamma_b \frac{h_i(R - \Sigma\tau_{bs})}{R - \Sigma\tau_{bs} + \frac{\tau_b}{4}}\right) T_{eff_i} +$$

$$\left(2\gamma_b \epsilon_i \sigma \frac{R - \Sigma\tau_{bs}}{R - \Sigma\tau_{bs} + \frac{\tau_b}{4}}\right) (T^4_{r_i} - T^4_{bs}) \tag{20}$$

Equating coefficients of T_{bs} to zero

$$0 = 1 - 2\gamma_b h_i \frac{R - \Sigma\tau_{bs}}{R - \Sigma\tau_{bs} + \frac{\tau_b}{4}} - 2\beta_b \frac{R - \Sigma\tau_{bs} + \frac{\tau_b}{2}}{R - \Sigma\tau_{bs} + \frac{\tau_b}{4}}$$

but $\gamma = \beta \frac{\tau}{k}$, thus

$$0 = 1 - 2\beta_b \frac{\tau_b}{k_b} h_i \frac{R - \Sigma\tau_{bs}}{R - \Sigma\tau_{bs} + \frac{\tau_b}{4}} - 2\beta_b \frac{R - \Sigma\tau_{bs} + \frac{\tau_b}{2}}{R - \Sigma\tau_{bs} + \frac{\tau_b}{4}}$$

Solving for β_b the critical β at the inner surface yields

$$\beta_b = \frac{1}{\left[\frac{2}{R - \Sigma\tau_{bs} + \frac{\tau_b}{4}}\right]\left[\left(R - \Sigma\tau_{bs} + \frac{\tau_b}{2}\right) + (R - \Sigma\tau_{bs})\frac{h_i\tau_b}{k_b}\right]} \tag{21}$$

For small τ_b's and small $\Sigma\tau_{bs}$'s, the one dimensional flat plate case (Equation 11) is approached as R becomes relatively large.

e. Observations

For finite τ's and $R \rightarrow \infty$, cylindrical temperature Equations (13), (16), (18), and (20) revert to flat-plate temperature Equations (3), (6), (8), and (10), respectively.

4. Transient, Radial Heat Transfer for Spheres

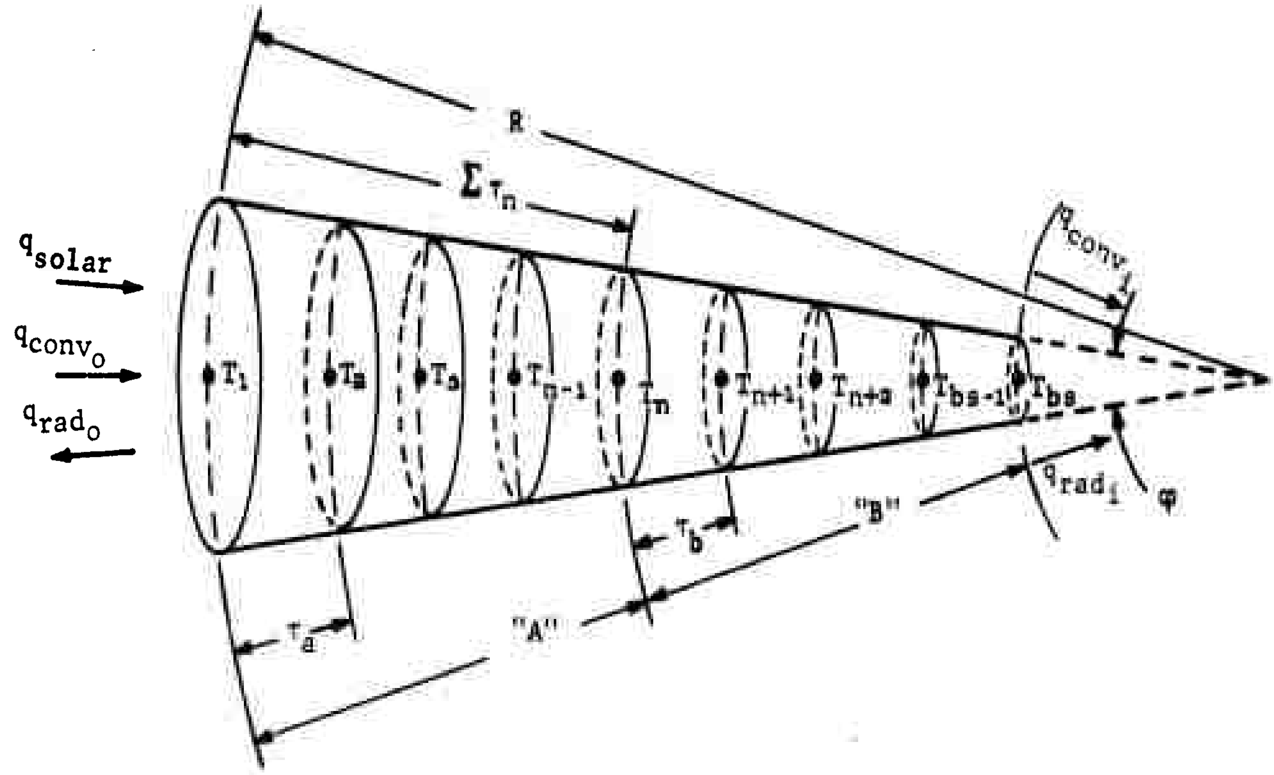

Figure 3

a. External Surface Equation

Consider a spherical segment heated as shown above. From the energy balance at the pheripheral surface

$$q_{conv_o} + q_{solar} - q_{rad_o} - q_{\underset{1\rightarrow 2}{cond}} = q_{\underset{1}{stored}} \tag{22}$$

or

$$\varphi R^2 h_o(T_{eff_o} - T_1) + \varphi R^2 q_{solar} - \varphi R^2 \epsilon_o\sigma(T_1^4 - T_{r_o}^4) -$$

$$\varphi\left[\left(R - \frac{\tau_a}{2}\right)^2 + \frac{\tau_a{}^2}{12}\right]\left[\frac{k_a(T_1 - T_2)}{\tau_a}\right] = \varphi\left[\left(R - \frac{\tau_a}{4}\right)^2 + \frac{\tau_a{}^2}{48}\right]\frac{\rho_a C_a \tau_a(T_1' - T_1)}{2\Delta T}$$

Note: See Appendix A for derivation of average areas.

Let $\beta = \dfrac{\Delta tk}{\rho C\tau^2}$ and $\gamma = \dfrac{\Delta t}{\rho C\tau}$

Then

$$T_1' = T_1\left[1 - 2\beta_a\frac{(1 - \tau_a/2R)^2 + \tau_a^2/12R^2}{(1 - \tau_a/4R)^2 + \tau_a^2/48R^2} - 2\gamma_a\frac{h_o}{(1 - \tau_a/4R)^2 + \tau_a^2/48R^2}\right] +$$

$$T_2\left[2\beta_a\frac{(1 - \tau_a/2R)^2 + \tau_a^2/12R^2}{(1 - \tau_a/4R)^2 + \tau_a^2/48R^2}\right] + T_{eff_o}\left[2\gamma_a\frac{h_o}{(1 - \tau_a/4R)^2 + \tau_a^2/48R^2}\right] +$$

$$q_{solar}\left[\frac{2\gamma_a}{(1 - \tau_a/4R)^2 + \tau_a^2/48R^2}\right] + \left(T_{r_o}^4 - T_1^4\right)\left[2\gamma_a\frac{\epsilon_o\sigma}{(1 - \tau_a/4R)^2 + \tau_a^2/48R^2}\right]$$

(23)

Set the coefficient of T_1 equal to zero and solve for the limiting β_a when h_o is a maximum.

$$\beta_a = \frac{1}{\left[\dfrac{2}{(1 - \tau_a/4R)^2 + \tau_a^2/48R^2}\right]\left[(1 - \tau_a/2R)^2 + \tau_a^2/12R^2 + \dfrac{h_o\tau_a}{k_a}\right]} \qquad (24)$$

For finite τ_a's Equation (24) approaches the flat-plate Equation (4) as R becomes large.

b. Interface Equation

At point T_n (interface of material "A" and "B") the energy balance is

$$q_{\underset{n-1\rightarrow n}{cond}} - q_{\underset{n\rightarrow n+1}{cond}} = q_{\underset{n}{stored}} \qquad (25)$$

or

$$\varphi \frac{k_a}{\tau_a}(T_{n-1} - T_n)\left[\left(R - \Sigma\tau_n + \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right] + \varphi\frac{k_b}{\tau_b}(T_{n+1} - T_n)\left[\left(R - \Sigma\tau_n - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right] =$$

$$\varphi\frac{(T_n' - T_n)}{2\Delta t}\left\{\rho_a C_a \tau_a\left[\left(R - \Sigma\tau_n + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right] + \rho_b C_b \tau_b\left[\left(R - \Sigma\tau_n - \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right]\right\}$$

Solve for T_n'

$$T_n' = T_n + (T_{n-1} - T_n)\left\{\frac{2\Delta t k_a\left[\left(R - \Sigma\tau_n + \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right]}{\tau_a\left\{\rho_a C_a \tau_a\left[\left(R - \Sigma\tau_n + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right] + \rho_b C_b \tau_b\left[\left(R - \Sigma\tau_n - \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right]\right\}}\right\}$$

$$(T_{n+1} - T_n)\left\{\frac{2\Delta t k_b\left[\left(R - \Sigma\tau_n - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right]}{\tau_b\left\{\rho_a C_a \tau_a\left[\left(R - \Sigma\tau_n + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right] + \rho_b C_b \tau_b\left[\left(R - \Sigma\tau_n - \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right]\right\}}\right\}$$

(26)

c. Interior Node Equation (Figure 3)

At point T_{n+1} (Material "B") the energy balance is

$$q_{\underset{n\rightarrow n+1}{cond}} - q_{\underset{n+1\rightarrow n+2}{cond}} = q_{\underset{n+1}{stored}} \qquad (27)$$

or

$$\varphi\left[\left(R - \Sigma\tau_{n+1} + \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right]\frac{k_b}{\tau_b}(T_n - T_{n+1}) + \varphi\left[\left(R - \Sigma\tau_{n+1} - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right]\frac{k_b}{\tau_b}(T_{n+2} - T_{n+1}) =$$

$$\varphi\left[\left(R - \Sigma\tau_{n+1}\right)^2 + \frac{\tau_b^2}{12}\right]\left(\frac{\rho_b C_b \tau_b}{\Delta t}\right)(T'_{n+1} - T_{n+1})$$

Let $\beta = \dfrac{\Delta t\, k}{\rho C \tau^2}$, then

$$T'_{n+1} = T_{n+1}\left\{1 - \beta_b \frac{\left[\left(R - \Sigma\tau_{n+1} + \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right] + \left[\left(R - \Sigma\tau_{n+1} - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right]}{\left(R - \Sigma\tau_{n+1}\right)^2 + \frac{\tau_b^2}{12}}\right\} +$$

$$\beta_b\left\{T_n \frac{\left(R - \Sigma\tau_{n+1} + \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}}{\left(R - \Sigma\tau_{n+1}\right)^2 + \frac{\tau_b^2}{12}} + T_{n+2}\frac{\left(R - \Sigma\tau_{n+1} - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}}{\left(R - \Sigma\tau_{n+1}\right)^2 + \frac{\tau_b^2}{12}}\right\}$$

(28)

d. Backside Surface Equation

At the backside surface (T_{bs}) the heat balance is

$$\underset{bs-1 \rightarrow bs}{q_{cond}} - q_{rad_i} - q_{conv_i} = \underset{bs}{q_{stored}} \qquad (29)$$

or

$$\varphi\frac{k_b}{\tau_b}(T_{bs-1} - T_{bs})\left[\left(R - \Sigma\tau_{bs} + \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right] + \varphi\,\epsilon_i\sigma(T_{r_i}^4 - T_{bs}^4)(R - \Sigma\tau_{bs})^2 +$$

$$\varphi\, h_i(T_{eff_i} - T_{bs})(R - \Sigma\tau_{bs})^2 = \varphi\left(\frac{\rho_b C_b \tau_b}{2\Delta t}\right)(T'_{bs} - T_{bs})\left[\left(R - \Sigma\tau_{bs} + \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right]$$

Let $\beta = \dfrac{\Delta t k}{\rho C \tau^2}$ and $\gamma = \dfrac{\Delta t}{\rho C \tau}$, then

$$T'_{bs} = T_{bs}\left\{1 - \frac{2\beta_b\left[\left(R - \sum\tau_{bs} + \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right]}{\left(R - \sum\tau_{bs} + \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}} - \frac{2\gamma_b h_i (R - \sum\tau_{bs})^2}{\left(R - \sum\tau_{bs} + \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}}\right\} +$$

$$T_{bs-1}\left\{\frac{2\beta_b\left[\left(R - \sum\tau_{bs} + \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right]}{\left(R - \sum\tau_{bs} + \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}}\right\} + T_{eff_i}\left\{\frac{2\gamma_b h_i (R - \sum\tau_{bs})^2}{\left(R - \sum\tau_{bs} + \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}}\right\} +$$

$$\left(T^4_{r_i} - T^4_{bs}\right)\left\{\frac{2\gamma_b \epsilon_i \sigma (R - \sum\tau_{bs})^2}{\left(R - \sum\tau_{bs} + \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}}\right\}$$

(30)

Equate the coefficient of T_{bs} to zero and solve for the limiting β_b when h_i is a maximum.

$$\beta_b = \frac{1}{\left[\frac{2}{\left(R - \sum\tau_{bs} + \frac{\tau_b}{4}\right)^2}\right]\left[\left(R - \sum\tau_{bs} + \frac{\tau_b}{2}\right)^2 + \frac{h_i\tau_b}{k_b}(R - \sum\tau_{bs})^2\right]} \tag{31}$$

As with the cylindrical Equation (21) this β_b approaches the flat-plate limits (Equation 11) when τ_b is finite and R is relatively large.

c. Observations

For finite τ's and $R \rightarrow \infty$, spherical temperature Equations (23), (26), (28), and (30) revert to the flat-plate temperature Equations (3), (6), (8), and (10), respectively.

5. General Equations Applicable to Flat Plates, Cylinders, and Spheres

Due to striking similarities between finite-difference heat transfer equations for flat plates, cylinders, and spheres, general equations adaptable to all three configurations are derived and presented as Equations (32) through (37). The coefficients G and m and the exponent m are inserted to perform manipulations resulting in equations applicable to the desired skin configuration. The following table gives values assigned to G and m.

Skin Configuration	G	m
Flat Plate	1	1
Cylinder	2	1
Sphere	2	2

Substitution of G = 1 and m = 1 into the general Equations (32) through (37) renders the flat-plate equations in Section II, Paragraph 2. Substitution of G = 2 and m = 1 into the general equations yields the cylindrical conduction equations of Section II, Paragraph 3. The spherical heat transfer equations of Section II, Paragraph 4, are obtained when G = 2 and m = 2 are placed in the general equations (Equations 32 through 37).

Since the cylindrical and spherical procedures have limiting β's (Equations 14, 21, 24, and 31) for the same reasons as the flat plate (Section II, Paragraph 2) and since the limiting β is a key factor in proper solution of the heat transfer equations the limits of this modulus are discussed in Appendix B for each basic skin configuration.

a. General External, Exposed Surface Equation (Any Homogeneous Material)

$$T_i' = T_i \left\{ 1 - 2\beta \frac{\left[1 - \frac{\tau}{2R}(G-1)\right]^m + \frac{\tau^m}{12R^m}(m-1)}{\left[1 - \frac{\tau}{4R}(G-1)\right]^m + \frac{\tau^m}{48R^m}(m-1)} - \frac{2\gamma h_o}{\left[1 - \frac{\tau}{4R}(G-1)\right]^m + \frac{\tau^m}{48R^m}(m-1)} \right\} +$$

$$T_s \left\{ 2\beta \frac{\left[1 - \frac{\tau}{2R}(G-1)\right]^m + \frac{\tau^m}{12R^m}(m-1)}{1 - \frac{\tau}{4R}(G-1)^m + \frac{\tau^m}{48R^m}(m-1)} \right\} - T_{eff_o} \left\{ \frac{2\gamma h_o}{\left[1 - \frac{\tau}{4R}(G-1)\right]^m + \frac{\tau^m}{48R^m}(m-1)} \right\} -$$

$$\left(T_{r_o}^4 - T_i^4\right) \left\{ \frac{2\gamma \epsilon_o \sigma}{\left[1 - \frac{\tau}{4R}(G-1)\right]^m - \frac{\tau^m}{48R^m}(m-1)} \right\} + q_{solar} \left\{ \frac{2\gamma}{\left[1 - \frac{\tau}{4R}(G-1)\right]^m + \frac{\tau^m}{48R^m}(m-1)} \right\} \tag{32}$$

b. General Interface Equation (Material "A" and "B")

$$T_n' = T_n + \left(T_{n-1} - T_n\right)\left\{\frac{2\Delta t k_a\left\{\left[R - \Sigma\tau_n + \frac{\tau_a}{2}(G-1)\right]^m + \frac{\tau_a^{\,m}}{12}(m-1)\right\}}{\tau_a\left[\rho_a C_a \tau_a\left\{\left[R - \Sigma\tau_n + \frac{\tau_a}{4}(m-1)\right]^m + \frac{\tau_a^{\,m}}{48}(m-1)\right\} + \rho_b C_b \tau_b\left\{\left[R - \Sigma\tau_n - \frac{\tau_b}{4}(m-1)\right]^m + \frac{\tau_b^{\,m}}{48}(m-1)\right\}\right]}\right\} +$$

$$\left(T_{n+1} - T_n\right)\left\{\frac{2\Delta t k_b\left\{\left[R - \Sigma\tau_n - \frac{\tau_b}{2}(G-1)\right]^m + \frac{\tau_b^{\,m}}{12}(m-1)\right\}}{\tau_b\left[\rho_a C_a \tau_a\left\{\left[R - \Sigma\tau_n + \frac{\tau_a}{4}(m-1)\right]^m + \frac{\tau_a^{\,m}}{48}(m-1)\right\} + \rho_b C_b \tau_b\left\{\left[R - \Sigma\tau_n - \frac{\tau_b}{4}(m-1)\right]^m + \frac{\tau_b^{\,m}}{48}(m-1)\right\}\right]}\right\} \tag{33}$$

c. General Interior Equation (Any Homogeneous Material)

$$T_x' = T_x\left\{1 - \beta\,\frac{\left\{\left[R - \Sigma\tau_x + \frac{\tau}{2}(G-1)\right]^m + \frac{\tau^m}{12}(m-1)\right\} + \left\{\left[R - \Sigma\tau_x - \frac{\tau}{2}(G-1)\right]^m + \frac{\tau^m}{12}(m-1)\right\}}{(R - \Sigma\tau_x)^m + \frac{\tau^m}{12}(m-1)}\right\} +$$

$$\beta\left\{T_{x-1}\,\frac{\left\{\left[R - \Sigma\tau_x + \frac{\tau}{2}(G-1)\right]^m + \frac{\tau^m}{12}(m-1)\right\}}{(R - \Sigma\tau_x)^m + \frac{\tau^m}{12}(m-1)} + T_{x+1}\,\frac{\left\{\left[R - \Sigma\tau_x - \frac{\tau}{2}(G-1)\right]^m + \frac{\tau^m}{12}(m-1)\right\}}{(R - \Sigma\tau_x)^m + \frac{\tau^m}{12}(m-1)}\right\} \tag{34}$$

d. General Backside Surface Equation (Any Homogeneous Material)

$$T'_{bs} = T_{bs}\left\{1 - 2\beta\,\frac{\left\{\left[R - \Sigma\tau_{bs} + \frac{\tau}{2}(G-1)\right]^m + \frac{\tau^m}{12}(m-1)\right\}}{\left\{\left[R - \Sigma\tau_{bs} + \frac{\tau}{4}(G-1)\right]^m + \frac{\tau^m}{48}(m-1)\right\}} - \frac{2\gamma h_i (R - \Sigma\tau_{bs})^m}{\left\{\left[R - \Sigma\tau_{bs} + \frac{\tau}{4}(G-1)\right]^m + \frac{\tau^m}{48}(m-1)\right\}}\right\} +$$

$$T_{bs-1}\left\{\frac{2\beta\left\{\left[R - \tau_{bs} + \frac{\tau}{2}(G-1)\right]^m + \frac{\tau^m}{12}(m-1)\right\}}{\left\{\left[R - \Sigma\tau_{bs} + \frac{\tau}{4}(G-1)\right]^m + \frac{\tau^m}{48}(m-1)\right\}}\right\} + T_{eff_i}\left\{\frac{2\gamma h_i (R - \Sigma\tau_{bs})^m}{\left\{\left[R - \Sigma\tau_{bs} + \frac{\tau}{4}(G-1)\right]^m + \frac{\tau^m}{48}(m-1)\right\}}\right\} +$$

$$\left(T_{r_i}^4 - T_{bs}^4\right)\left\{\frac{2\gamma\epsilon_i\sigma(R - \Sigma\tau_{bs})^m}{\left\{\left[R - \Sigma\tau_{bs} + \frac{\tau}{4}(G-1)\right]^m + \frac{\tau^m}{48}(m-1)\right\}}\right\} \tag{35}$$

e. General Limiting Criteria

(1) External, Exposed Surface (Any Homogeneous Material)

$$\beta = \frac{1}{\dfrac{2}{\left\{\left[1 - \frac{\tau}{4R}(G-1)\right]^m + \frac{\tau^m}{48R}m(m-1)\right\}}\left\{\left\{\left[1 - \frac{\tau}{2R}(G-1)\right]^m + \frac{\tau^m}{12R}m(m-1)\right\} + \frac{h_o\tau}{k}\right\}} \tag{36}$$

(2) Internal Surface or Center (Any Homogeneous Material)

$$\beta = \frac{1}{\dfrac{2}{\left\{\left[R - \Sigma\tau_{bs} + \frac{\tau}{4}(G-1)\right]^m + \frac{\tau^m}{48}(m-1)\right\}}\left\{\left\{\left[R - \Sigma\tau_{bs} + \frac{\tau}{2}(G-1)\right]^m + \frac{\tau^m}{12}(m-1)\right\} + (R - \Sigma\tau_{bs})^m\frac{h_i\tau}{k}\right\}} \tag{37}$$

6. Finite-Difference Results and Procedure Selection

Calculated results from the three finite-difference heat transfer procedures presented in this report have been compared with the exact solutions.[7,8] Excellent agreement, within 0.3 percent, was obtained in each case for a material initially at uniform temperature (T_0) and with one of its surfaces maintained at a constant temperature different from T_0 after time zero. The accuracy of any finite-difference heat flow equation depends on the proper selection of the time interval (Δt) and distance increment (τ) as well as the closeness of the approximate temperature gradients, $\frac{T_x - T_{x+1}}{\tau}$, to the true gradients.

For the majority of missile structure analysis, the simple flat-plate procedure may be sufficiently accurate. For instance, examination of the cylindrical and spherical conduction equations (Equations 20 and 30) at the inside surface (where the maximum deviation from a flat plate obviously will occur) shows that a ratio of $R/\Sigma\tau_{bs}$ (radius/wall thickness) of 100 gives results approximating, within 0.6 percent, the flat-plate solution (Equation 10). Thus, for adequate engineering analyses and structural design, the simple one-dimensional flat-plate procedure is advantageous in many investigations because of the required computer time.

Section III. CONCLUSIONS

One general finite-difference heat-transfer procedure can be used with minor manipulations, to calculate transient, radial heat flow in spheres and cylinders and to calculate one-dimensional heat flow in flat plates.

Excellent agreement between finite-difference calculations and exact analytical results is obtained when proper time and distance increments are chosen in relation to material thermal properties.

For hollow or solid spheres and cylinders composed of one homogeneous material, the limiting modulus, β, may be determined by parameters at the internal surface even though the external surface may be the only surface exposed to convective heating.

Comparisons of finite-difference calculations with exact analytical results for several special heat transfer cases are necessary for final determination of the accuracy of the finite-difference methods. Results of these additional comparisons will be published in a future document.

LITERATURE CITED

1. N. J. Hurst, Measurements of Heating to a Thermally Thick Spherical Copper Calorimeter, ABMA Report RSM-TM-1-60, October 1960.

2. W. B. Scholten, Evaluation of the Thermal Performance of the Ablation Materials on Recovered Jupiter IRBM Reentry Nose Cones (U). ABMA Report DSD-TR-6-60 (Secret), January 1960.

3. W. G. Burleson, Comparison of Calculated and Inflight Measured Data for Metallic and Fiber Reinforced Materials on Hypersonic Vehicles (U), ABMA Report DSD-TM-10-60 (Secret), June 1960.

4. N. J. Hurst, Calculations of the Simultaneous Ablation Procession and Associated Temperature Distribution in a Heterogeneous Wall Exposed to a Time-Variable Heat Flux, ABMA Internal Note #2, February 1960.

5. B. M. Kavanaugh, Aerodynamic Heating Analysis of a Tactical Pershing Reentry Body Having a Reduced Insulation Thickness (U), USAMICOM Report RS-TM-63-4 (Confidential), October 1963.

6. W. G. Burleson, et al, Thermal Analysis of a Pershing XB-1 Motor Subjected to a Cold Environment (U), USAMICOM Report RS-TR-62-3 (Confidential), November 1962.

7. H. S. Carslaw and J. C. Jaeger, Conduction of Heat in Solids, Second edition, Oxford University Press, 1959.

8. M. Jakob, Heat Transfer, Volume I, John Wiley & Sons, Inc., 1959.

SELECTED BIBLIOGRAPHY

1. Heat-Transfer Calculations by Finite Differences, G. M. Dusinberre, 1961, International Textbook Company.

2. "Numerical Methods for Transient Heat Flow," G. M. Dusinberre, Transactions of the ASME, November 1945.

3. Graphical Presentation of Difference Solutions for Transient Radial Heat Conduction in Hollow Cylinders with Heat Transfer at the Inner Radius and Finite Slabs with Heat Transfer at One Boundary, J. E. Hatch, et al., NASA TR R-56, 1960.

4. The Calculation of Transient Temperature Distributions in a Solid Cylindrical Pin, Cooled on the Surface, J. Heestand, D. F. Schoeberle, and L. B. Miller, Argonne/National Laboratory Report 6237, October 1960.

5. The Solution of Transient Heat Conduction Problems by Finite Differences, Purdue University /EES RS 98, March 1947.

6. J. E. Medford, "Transient Radial Heat Transfer in Uncooled Rocket Nozzles," Aerospace/Engineering, Vol. 21, No. 6, October 1962.

7. G. R. Gaumer, "Stability of Three Finite Difference Methods of Solving for Transient Temperatures," ARS Journal, Vol. 32, No. 10, October 1962.

8. A. G. Elrod, Jr., "New Finite-Difference Technique for Solution of the Heat-Conduction Equation, Especially Near Surfaces with Convective Heat Transfer," Transactions of the ASME, Vol. 79, 1957.

9. R. D. Geckler, "Transient, Radial Heat Conduction in Hollow Cylinders," Jet Propulsion, Vol. 25, January 1955.

Appendix A

DERIVATION OF AVERAGE AREA EQUATIONS FOR SPHERICAL CONDUCTION

Since the surface area subtended by a solid angle on a sphere is not a linear function of the radius (R) the average area ($\bar{A}$) through which heat flows and the average area for determining heat storage volume ($\tau\bar{A}$) must be found.

1. External Surface

The average area from 1 to 2 (Figure 3) is found from

$$\bar{A} = \frac{\varphi \int_{R-\tau_a}^{R} R^2 dR}{\tau_a} = \varphi\left[\left(R - \frac{\tau_a}{2}\right)^2 + \frac{\tau^2}{12}\right] \tag{38}$$

To find the average area for determining the heat storage volume from R to $R - \frac{\tau_a}{2}$ the following equation is used.

$$\bar{A} = \frac{\varphi \int_{R-\tau_a/2}^{R} R^2 dR}{\tau_a/2} = \varphi\left[\left(R - \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right] \tag{39}$$

Results of the integration in Equations (38) and (39) are reflected in Equation (22) of the text.

2. Interface

The average area from n-1 to n (Figure 3) is

$$\bar{A} = \frac{\varphi \int_{R-\sum\tau_n}^{R-\sum\tau_n+\tau_a} R^2 dR}{\tau_a} = \varphi\left[\left(R - \sum\tau_n + \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right] \tag{40}$$

and the average area from n to n+1 is

$$\bar{A} = \frac{\varphi \int_{R-\Sigma\tau_n-\tau_b}^{R-\Sigma\tau_n} R^2 dR}{\tau_b} = \varphi\left[\left(R - \Sigma\tau_n - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right] \tag{41}$$

The average area for finding the heat storage volume at an interface (n) is

$$\bar{A} = \frac{\varphi \int_{R-\Sigma\tau_n-\tau_b/2}^{R-\Sigma\tau_n+\tau_a/2} R^2 dR}{\frac{\tau_a + \tau_b}{2}} = \varphi\left\{\left[\left(R - \Sigma\tau_n + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right] + \left[\left(R - \Sigma\tau_n - \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right]\right\} \tag{42}$$

Results of the integration of Equations (40), (41), and (42) are used in Equations (26).

3. Interior

The average area through which heat flows from n+1 to n+2 is given by Equation (43). The $\bar{A}$ between n and n+1 is found by using $+ \tau_b/2$ in Equation (43).

$$\bar{A} = \frac{\varphi \int_{R-\Sigma\tau_{n+1}-\tau_b}^{R-\Sigma\tau_{n+1}} R^2 dR}{\tau_b} = \varphi\left[\left(R - \Sigma\tau_{n+1} - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right] \tag{43}$$

To obtain the heat storage volume at node n+1 the following average area is used.

$$\bar{A} = \frac{\varphi \int_{R-\Sigma\tau_{n+1}-\tau_b/2}^{R-\Sigma\tau_{n+1}+\tau_b/2} R^2 dR}{\tau_b} = \varphi\left[\left(R - \Sigma\tau_{n+1}\right)^2 + \frac{\tau_b^2}{12}\right] \tag{44}$$

Results of the integration in Equations (43) and (44) are inserted in the interior Equation (27).

4. **Backside Surface**

The average area through which heat is conducted from nodes bs-1 to bs (Figure 3) is

$$\bar{A} = \frac{\varphi \int_{R-\Sigma\tau_{bs}}^{R-\Sigma\tau_{bs}+\tau_b} R^2 dR}{\tau_b} = \varphi\left[\left(R - \Sigma\tau_{bs} + \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right] \qquad (45)$$

and the area determining the heat storage volume is

$$\bar{A} = \frac{\varphi \int_{R-\Sigma\tau_{bs}}^{R-\Sigma\tau_{bs}+\tau_b/2} R^2 dR}{\frac{\tau_b}{2}} = \varphi\left[\left(R - \Sigma\tau_{bs} + \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right] \qquad (46)$$

The results of the integrations of Equations (45) and (46) are reflected in Equation (29).

Appendix B

LIMITS OF THE DIMENSIONLESS MODULUS, β, FOR FLAT PLATES, CYLINDERS, AND SPHERES

In all conduction equations presented in this report, the modulus, β, has limits which are critical to the stability of the equations. This is due to the necessity of coefficients of a particular temperature point being equal to or greater than zero because it is unreasonable for a temperature to depend on a previous temperature in the negative sense. The following equations show the limits of β under conditions where the heat transfer coefficient (h) equals zero.

1. Heat Transfer Coefficient (h) = 0

a. Flat Plate

The equations of β applicable to the external and internal surface (Equations 4 and 11) reduce to the following upper limit when $h_o = 0$ and $h_i = 0$.

$$\beta \leq \frac{1}{2} \tag{47}$$

b. Cylinder

(1) External Surface

The limiting β equation (Equation 14) at the external surfaces reduces to the following when $h_o = 0$.

$$\beta \leq \frac{1}{2\left(\frac{R - \tau_a/2}{R - \tau_a/4}\right)} \tag{48}$$

which has values approaching 0.5 and 0.75 for $R \gg \tau_a$ and $\tau_a \rightarrow R$ respectively.

(2) Internal Surface

The limiting β at the internal surface or center of a cylinder (Equation 21) reduces to the following when $h_i = 0$.

$$\beta_b \leq \frac{1}{2\left(\frac{R - \sum\tau_{bs} + \tau_b/2}{R - \sum\tau_{bs} + \tau_b/4}\right)} \tag{49}$$

The values of Equation (49) approach 0.25 and 0.5 as $\Sigma\tau_{bs} \rightarrow R$ and when $R \gg \Sigma\tau_{bs}$ respectively. Thus when $h_o = 0$ the limiting modulus β of a cylinder composed of one homogeneous material may be determined by Equation (49) rather than Equation (48). The limiting β varies radically from 0.5 only when $R - \Sigma\tau_{bs}$ is less than one τ.

c. Sphere

(1) External Surface

The critical β in Equation (24) reduces to the following when $h_o = 0$.

$$\beta_a \leq \frac{1}{2\left[\dfrac{\left(1 - \dfrac{\tau_a}{2R}\right)^2 + \dfrac{\tau_a^2}{12R^2}}{\left(1 - \dfrac{\tau_a}{4R}\right)^2 + \dfrac{\tau_a^2}{48R^2}}\right]} \qquad (50)$$

The values of β in this equation approach 0.5 and 0.875 when $R \gg \tau_a$ and when $\tau_a \rightarrow R$ respectively.

(2) Internal Surface

The limiting β in Equation (31) reduces to the following when $h_i = 0$.

$$\beta_b \leq \frac{1}{2\left[\dfrac{\left(R - \Sigma\tau_{bs} + \dfrac{\tau_b}{2}\right)^2}{\left(R - \Sigma\tau_{bs} + \dfrac{\tau_b}{4}\right)^2}\right]} \qquad (51)$$

The β in this equation has values approaching 0.125 and 0.5 when $\Sigma\tau_{bs} \rightarrow R$ and when $R \gg \Sigma\tau_{bs}$ respectively. Thus the overall critical β for solid or near solid spheres composed of one homogeneous material may be determined by Equation (51) rather than Equation (50). The lower β limit of 0.125 is approached rapidly after $R - \Sigma\tau_{bs}$ becomes less than one τ.

2. Heat Transfer Coefficient $\neq 0$

To determine the limits of β for each of the three configurations when $h \neq 0$ the β must be determined for the maximum heat transfer coefficient. Both external and internal surfaces β's should be calculated to determine upper limits which are valuable in insuring stability or convergence of the equations.

DISTRIBUTION

	No. of Copies
U. S. Army Missile Command Distribution List A	82
Defense Documentation Center Cameron Station Alexandria, Virginia - 22314	20
Director Advanced Research Projects Agency Attn: Mr. John Crone The Pentagon, Room 2B260 Washington, D. C. 20301	1
MIT, Lincoln Laboratory Attn: Dr. Melvin Dill P.O. Box 73 Lexington 73, Massachusetts 02173	1
AMCPM-MA	1
-MB	1
-PE	1
-SE	1
-ZE	1
AMSMI-R, Mr. McDaniel	1
-RAP	1
-RBL	5
-RD	1
-RE	1
-RF	1
-RG	1
-RH	1
-RHT, Mr. Garner	1
-RK	1
-RL, Mr. Bounds	1
-RNT, Mr. Dobbins	1
-RR	1
-RR, Mr. J. Bennett	1
-RS	1
-RSS	1
-RST	30
-RT	1
-Y, Mr. J. Williams	1

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1 ORIGINATING ACTIVITY (Corporate author) Structures and Mechanics Laboratory Directorate of Research and Development U. S. Army Missile Command Redstone Arsenal, Alabama	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b GROUP N/A

3. REPORT TITLE

SOLUTION OF TRANSIENT HEAT TRANSFER PROBLEMS FOR FLAT PLATES, CYLINDERS, AND SPHERES BY FINITE-DIFFERENCE METHODS

4 DESCRIPTIVE NOTES (Type of report and inclusive dates)

Progress Report

5 AUTHOR(S) (Last name, first name, initial)

Burleson, W. G. and Eppes, R., Jr.

6 REPORT DATE	7a TOTAL NO OF PAGES	7b. NO OF REFS
15 March 1965	29	8

8a CONTRACT OR GRANT NO	9a ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO (DA) 1A013001A039	RS-TR-65-1
c. AMC Management Structure Code No. 501611844 d	9b OTHER REPORT NO(S) (Any other numbers that may be assigned this report) AD________

10 AVAILABILITY/LIMITATION NOTICES

Qualified requesters may obtain copies of this report from DDC.

11 SUPPLEMENTARY NOTES	12 SPONSORING MILITARY ACTIVITY
None	Same as No. 1

13 ABSTRACT

Presented in this report are finite-difference (forward) heat transfer equations applicable to transient, radial heat flow in spheres and cylinders and applicable to transient, one-dimensional heat flow in flat plates. By very minor manipulations, a single heat transfer procedure can easily be utilized to determine transient heat flow in all three basic structure configurations.

For each skin configuration the accuracy of the finite-difference procedure, compared with exact analytical methods, depends on optimum selection of the calculation time increment and the incremental distance between temperature nodes in relation to the material thermal properties and on the closeness of the approximate temperature gradients to the true gradients.

DD FORM 1 JAN 64 1473

UNCLASSIFIED
Security Classification

14. (U) KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
Finite-Difference						
Heat Transfer						
Heat Condition						
Radial Heat Flow						

INSTRUCTIONS

1. **ORIGINATING ACTIVITY:** Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2*a*. **REPORT SECURITY CLASSIFICATION:** Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2*b*. **GROUP:** Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. **REPORT TITLE:** Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. **DESCRIPTIVE NOTES:** If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. **AUTHOR(S):** Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. **REPORT DATE:** Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7*a*. **TOTAL NUMBER OF PAGES:** The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7*b*. **NUMBER OF REFERENCES:** Enter the total number of references cited in the report.

8*a*. **CONTRACT OR GRANT NUMBER:** If appropriate, enter the applicable number of the contract or grant under which the report was written.

8*b*, 8*c*, & 8*d*. **PROJECT NUMBER:** Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9*a*. **ORIGINATOR'S REPORT NUMBER(S):** Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9*b*. **OTHER REPORT NUMBER(S):** If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. **AVAILABILITY/LIMITATION NOTICES:** Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through

____________________________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through

____________________________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through

____________________________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. **SUPPLEMENTARY NOTES:** Use for additional explanatory notes.

12. **SPONSORING MILITARY ACTIVITY:** Enter the name of the departmental project office or laboratory sponsoring *(pa[illegible] ing for)* the research and development. Include address.

13. **ABSTRACT:** Enter an abstract giving a brief and [illegible] summary of the document indicative of the report, e[illegible] though it may also appear elsewhere in the body of the te[illegible] report. If additional space is required, a continua[illegible] sheet shall be attached.

It is highly desirable that the abstract [illegible] classified reports be unclassified. Each paragraph of [illegible] abstract shall end with an indication of the military [illegible] classification of the information in the paragraph, re[illegible] sented as (TS) (S), (C), or (U).

There is no limitation on the [illegible] of the abstract. However, the suggested length is fro[illegible] 150 to 225 words.

14. **KEY WORDS:** Key words [illegible] technically meaningful terms or short phrases that chara[illegible] a report and may be used as index entries for catalogi[illegible] the report. Key words must be selected so that no secu[illegible] classification is required. Identifiers, such as equipmen[illegible] model designation, trade name, military project code name[illegible] geographic location, may be used as key words but will b[illegible] followed by an indication of technical context. The assi[illegible] of links, rules, and weights is optional.